

# ONE-PARAMETER KOOPMAN SEMIGROUPS ON $L^p$ - AND $C(K)$ -SPACES

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*Dedicated to Rainer Nagel on the occasion of his 75th birthday.*

ABSTRACT. We characterize Markov lattice semigroups induced by measurable semiflows on probability spaces by properties of their generators. In addition we construct *topological models* on compact spaces for such semigroups.

## 1. INTRODUCTION

It is our goal to characterize Markov lattice semigroups  $(T(t))_{t \geq 0}$  on Banach lattices  $L^p(X, \Sigma, \mu)$ ,  $1 \leq p < \infty$ , by means of their generator  $A$ . In our main result 2.1 we show that the generator  $A$  is a derivation on a suitable subspace of  $L^\infty(X)$ . We also show that such semigroups  $(T(t))_{t \geq 0}$  are induced by measurable semiflows, i.e., each operator  $T(t)$  is of the form

$$T(t)f = f \circ \varphi_t$$

for all  $f \in L^p(X)$ ,  $t \geq 0$  and measurable measure-preserving self-mappings  $\varphi_t: X \rightarrow X$ . Such operators are called Koopman operators or composition operators. This work was motivated by the manuscript [6] of T. ter Elst and M. Lemańczyk and the results in [4] by R. Derndinger and R. Nagel and in [2], B-II, Section 3. We refer to [3] for the significance of these semigroups for application.

In the next step, we give a similar characterization for lattice semigroups that are not necessarily Markov. Then we use Koopman semigroups to construct a topological model on a compact space for a measurable semiflow. Our proof follows [5], Chapter 12. If the underlying probability space  $(X, \Sigma, \mu)$  is separable, then the compact space can even be chosen to be metrizable. This is a result by W. Ambrose and S. Kakutani, [1], Theorem 5. In the last part, we discuss ergodic semiflows and give a new proof for Theorem 1 in [9] using results from the Perron Frobenius spectral theory.

We now establish concepts and notations used in this paper. We start from a probability space  $X := (X, \Sigma, \mu)$  and the corresponding Banach space  $L^p(X; \mathbb{C})$ ,  $1 \leq p \leq \infty$ , with the usual norm and write shortly  $L^p(X)$ . These spaces are even Banach lattices. A positive semigroup  $(T(t))_{t \geq 0}$  on  $L^p(X)$ ,  $1 \leq p < \infty$ , is called a

lattice semigroup if each operator  $T(t)$  satisfies

$$|T(t)f| = T(t)|f|$$

for all  $f \in L^p(X)$ . We call such a semigroup Markov if

$$T(t)\mathbb{1} = \mathbb{1} \quad \text{and} \quad \int_X T(t)f \, d\mu = \int_X f \, d\mu$$

for all  $f \in L^p(X)$ ,  $t \geq 0$ . For the second equation we write shortly  $T(t)'\mu = \mu$ . It follows that  $\|T(t)f\|_1 = \|f\|_1$  for all  $f \in L^p(X)$ ,  $t \geq 0$ , hence each Markov operator on  $L^p(X)$  extends to a Markov operator on  $L^1(X)$ . Additionally, we need the following definitions. A family  $(\varphi_t)_{t \geq 0}$  of self-mappings on a probability space  $(X, \Sigma, \mu)$  is called a measurable semiflow if  $\varphi_0 = \text{id}_X$ ,  $\varphi_{s+t} = \varphi_s \circ \varphi_t$  for all  $s, t \geq 0$  and every  $\varphi_t$  is measurable and measure-preserving, i.e.  $\mu(\varphi_t^{-1}(A)) = \mu(A)$  for all  $A \in \Sigma$ . We then consider the equivalence relation

$$M \sim N : \Longleftrightarrow \mathbb{1}_M = \mathbb{1}_N \quad \mu\text{-almost everywhere}$$

on  $\Sigma$ . This yields the measure algebra  $\Sigma(X) := \Sigma / \sim$ . Each  $\varphi_t$  induces a mapping

$$\varphi_t^* : \Sigma(X) \rightarrow \Sigma(X), \quad [M] \mapsto [\varphi_t^{-1}(M)].$$

For  $N \in \Sigma$  we denote by  $\mathbb{1}_{[N]}$  the equivalence class  $[\mathbb{1}_N]$ . For further information on  $\Sigma(X)$  we refer to [5], Chapter 6. A mapping  $\theta : \Sigma(X) \rightarrow \Sigma(X)$  is called a measure algebra homomorphism if it respects both the order on  $\Sigma(X)$  and the measure  $\mu$ . Each measure algebra homomorphism corresponds uniquely to a Markov lattice operator  $T$  on  $L^1(X)$  by  $T\mathbb{1}_M = \mathbb{1}_{\theta(M)}$ ,  $M \in \Sigma(X)$ , see [5], Theorem 12.10.

Finally we call an operator  $\delta$  on  $L^\infty(X)$  or on the space  $C(K)$  of all complex-valued continuous functions on a compact Hausdorff space  $K$  a *derivation* if its domain  $D(\delta)$  is a conjugation invariant subalgebra with  $\mathbb{1} \in D(\delta)$  such that  $\delta(f \cdot g) = \delta f \cdot g + f \cdot \delta g$  and  $\delta \bar{f} = \overline{\delta f}$  for all  $f, g \in D(\delta)$ .

## 2. CHARACTERIZATION OF KOOPMAN SEMIGROUPS ON $L^p$ -SPACES

As a first result we show that a  $C_0$ -semigroup of Markov lattice operator is induced by a semiflow on the measure algebra and that its generator  $A$  is a derivation restricted to  $D(A) \cap L^\infty(X)$ .

**Theorem 2.1.** *Let  $(X, \Sigma, \mu)$  be a probability space and  $(T(t))_{t \geq 0}$  a  $C_0$ -semigroup on  $L^1(X)$  with generator  $(A, D(A))$ . Then the following are equivalent.*

- (i)  $(T(t))_{t \geq 0}$  is a semigroup of Markov lattice operators.
- (ii) For each  $t \geq 0$  there exists a measure algebra homomorphism  $\varphi_t^* : \Sigma(X) \rightarrow \Sigma(X)$  such that  $T(t)\mathbb{1}_M = \mathbb{1}_{\varphi_t^*(M)}$  for all  $M \in \Sigma(X)$ .
- (iii) The space  $L^\infty(X)$  is invariant under  $(T(t))_{t \geq 0}$ ,  $D(A) \cap L^\infty(X)$  is a dense algebra in  $L^1(X)$  and  $A$  is a derivation on this algebra with  $A'\mu = 0$ .

*Proof.* The equivalence (i)  $\Leftrightarrow$  (ii) is given by [5], Theorem 12.10, where it is shown that every Markov lattice operator  $T(t)$  corresponds to a measure algebra homomorphism  $\varphi_t^*$  via  $T(t)\mathbb{1}_M = \mathbb{1}_{\varphi_t^*(M)}$ .

To prove the implication (ii)  $\Rightarrow$  (iii), we remark that (ii) implies that each  $T(t)$  leaves the real part invariant and is also multiplicative since

$$T(t)(\mathbb{1}_M \cdot \mathbb{1}_N) = T(t)\mathbb{1}_{M \cap N} = \mathbb{1}_{\varphi_t^*(M \cap N)} = T(t)\mathbb{1}_M \cdot T(t)\mathbb{1}_N$$

for  $M, N \in \Sigma(X)$ ,  $t \geq 0$ . This extends to  $L^\infty(X)$  by linearity and density. Since  $T(t)\mathbb{1} = \mathbb{1}$  for all  $t \geq 0$ , we obtain  $\mathbb{1} \in D(A)$ .

For  $f, g \in D(A) \cap L^\infty(X)$  we consider the mapping  $t \mapsto T(t)(f \cdot g)$ . Using the multiplicativity of  $T(t)$  and differentiation with respect to  $\|\cdot\|_1$  yields

$$\begin{aligned} \frac{d}{dt}T(t)(f \cdot g) &= \frac{d}{dt}(T(t)f \cdot T(t)g) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} ([T(t+h)f - T(t)f]T(t)g \\ &\quad + T(t+h)f[T(t+h)g - T(t)g]) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} ([T(t+h)f - T(t)f]T(t)g) \\ &\quad + \lim_{h \rightarrow 0} T(t+h)f \cdot \lim_{h \rightarrow 0} \frac{1}{h} [T(t+h)g - T(t)g] \\ &= \left( \frac{d}{dt}T(t)f \right) \cdot T(t)g + T(t)f \cdot \left( \frac{d}{dt}T(t)g \right), \end{aligned}$$

where we used that multiplication by a bounded function is a continuous operator on  $L^1(X)$ . For  $t = 0$  we obtain  $A(f \cdot g) = Af \cdot g + f \cdot Ag$ , hence  $f \cdot g \in D(A) \cap L^\infty(X)$  and  $A$  is a derivation. Since each  $T(t)$  is an isometry on  $L^1(X)$ , it follows that  $T(t)'\mu = \mu$  for each  $t \geq 0$ , hence  $A'\mu = 0$ . Since  $D(A) \cap L^\infty(X)$  is a dense sublattice of  $D(A)$ , it is dense in  $L^1(X)$ .

To prove (iii)  $\Rightarrow$  (i) we first show that  $T(t)$  is multiplicative on  $L^\infty(X)$ . For  $f, g \in D(A) \cap L^\infty(X)$  we have  $f \cdot g \in D(A) \cap L^\infty(X)$  by assumption. Fix  $t > 0$  and consider the map  $s \mapsto \eta(s) := T(t-s)[T(s)f \cdot T(s)g]$  for  $0 \leq s \leq t$ . Obviously  $\eta(0) = T(t)(f \cdot g)$  and  $\eta(t) = T(t)f \cdot T(t)g$ . If we set  $P(\cdot) := T(t-\cdot)$  and  $Q(\cdot) := T(\cdot)$ , we can apply Lemma B.16 in [7] to  $\eta(\cdot)$ . Since  $A$  is a derivation, we obtain  $\eta'(\cdot) \equiv 0$ , hence  $\eta$  is constant. So  $T(t)$  is multiplicative on  $D(A) \cap L^\infty(X)$  and, since  $D(A) \cap L^\infty(X)$  is  $\|\cdot\|_1$ -dense, on  $L^\infty(X)$ .

Clearly  $T(t)\overline{f} = \overline{T(t)f}$  since  $A$  is a real operator. The multiplicativity of  $T(t)$  implies that each  $T(t)$  is a lattice homomorphism (cf. [5], Theorem 7.23). For  $f \in L^\infty(X, \mathbb{R})$  and  $t \geq 0$  we obtain  $T(t)(f^2) = T(t)f \cdot T(t)f \geq 0$ , hence  $(T(t))_{t \geq 0}$  is a positive  $C_0$ -semigroup. The assumptions  $A'\mu = 0$  and  $A\mathbb{1} = 0$  imply  $T(t)'\mu = \mu$  and  $T(t)\mathbb{1} = \mathbb{1}$ . Hence we have shown that  $(T(t))_{t \geq 0}$  is a  $C_0$ -semigroup of Markov lattice operators.  $\square$

As a consequence, we obtain that on standard probability spaces the semigroups appearing above are indeed Koopman semigroups, i.e., induced by a measurable semiflow on the probability space.

**Corollary 2.2.** *If, in addition,  $(X, \Sigma, \mu)$  is a standard probability space, then (i), (ii) and (iii) of 2.1 are equivalent to*

- (iv) *There exists a measurable semiflow  $(\varphi_t)_{t \geq 0}$  such that  $T(t)f = f \circ \varphi_t$  for all  $f \in L^1(X)$ ,  $t \geq 0$ .*

*Proof.* If (i) in 2.1 holds, we obtain the assertion by von Neumann's Theorem (cf. [5], Appendix F.3). On the other hand, since each  $\varphi_t$  is measure-preserving, we obtain measure algebra homomorphisms  $\varphi_t^*: \Sigma(X) \rightarrow \Sigma(X)$  such that  $T(t)\mathbb{1}_M = \mathbb{1}_M \circ \varphi_t = \mathbb{1}_{\varphi_t^{-1}(M)} = \mathbb{1}_{\varphi_t^*(M)}$  for all  $M \in \Sigma(X)$  and  $t \geq 0$ .  $\square$

We remark that Theorem 1.1 in [6] can be obtained by the following corollary.

**Corollary 2.3.** *The equivalences in 2.1 hold for  $C_0$ -semigroups on each  $L^p(X)$ ,  $1 \leq p < \infty$ .*

In the next theorem we extend the above characterization to one-parameter lattice semigroups on  $L^p$ -spaces (not necessarily being Markov).

**Theorem 2.4.** *Let  $(X, \Sigma, \mu)$  be a probability space and  $(S(t))_{t \geq 0}$  a  $C_0$ -semigroup on  $L^1(X)$  with generator  $A$ ,  $\mathbb{1} \in D(A)$  and assume  $q := A\mathbb{1} \in L^\infty(X)$ . Then the following are equivalent.*

- (i)  *$(S(t))_{t \geq 0}$  is a lattice semigroup.*
- (ii) *There exists a derivation  $\delta$  on the algebra  $D(\delta) \cap L^\infty(X)$  such that  $A = \delta + q$  (i.e.,  $D(A) = D(\delta)$  and  $Af = \delta f + qf$  for all  $f \in D(A)$ .)*

Moreover, if (i) and (ii) hold, then  $(S(t))_{t \geq 0}$  is given by

$$S(t)f = \exp \left( \int_0^t T(s)q \, ds \right) T(t)f$$

for all  $f \in L^1(X)$ ,  $t \geq 0$ , where  $(T(t))_{t \geq 0}$  is the Markov lattice semigroup generated by  $\delta$ .

*Proof.* To show the equivalence of (i) and (ii), we recall from [2], Section C-II Corollary 5.8, or [8], Theorem 3.4, that  $A$  generates a lattice semigroup on  $L^1(X)$  if and only if it is a real operator and satisfies the *Kato equality*, i.e.,  $f \in D(A)$  implies  $|f| \in D(A)$  and  $A|f| = \text{sign}(f)Af$  for all real-valued  $f \in D(A)$ . Since  $\delta$  is a perturbation of  $A$  by a bounded multiplication operator, it suffices to observe that  $A$  satisfies the Kato equality if and only if  $\delta = A - q$  does.

Now assume that (i) and (ii) hold. Define operators  $S(t)$  by

$$S(t)f := \exp \left( \int_0^t T(s)q \, ds \right) T(t)f,$$

$f \in L^1(X)$ . A short computation shows that  $(S(t))_{t \geq 0}$  is again a  $C_0$ -semigroup. Let  $B$  denote its generator. Differentiation with respect to  $\|\cdot\|_1$  yields

$$\left. \frac{d}{dt} \right|_{t=0} S(t)f = qf + \delta f$$

for all  $f \in D(B)$ . Therefore,  $\delta + q \supset B$ , thus  $\delta + q = B$  since  $\delta + q$  is a generator itself.  $\square$

**Corollary 2.5.** *If, in addition to the assumptions in 2.4,  $(X, \Sigma, \mu)$  is a standard probability space, then (i) and (ii) above are equivalent to*

- (iii) *There exists a measurable semiflow  $(\varphi_t)_{t \geq 0}$  such that*

$$S(t)f = \exp \left( \int_0^t q \circ \varphi_s \, ds \right) \cdot f \circ \varphi_t$$

*for all  $f \in L^1(X)$  and  $q = A\mathbf{1}$ .*

*Proof.* This follows directly from 2.1.  $\square$

### 3. TOPOLOGICAL MODEL

On standard probability spaces  $(X, \Sigma, \mu)$ , 2.5 shows that the semigroup  $(T(t))_{t \geq 0}$  is induced by a measurable semiflow  $(\varphi_t)_{t \geq 0}$  on  $X$  and we call  $(T(t))_{t \geq 0}$  the Koopman semigroup induced by  $(\varphi_t)_{t \geq 0}$ . We now show that this situation can always be achieved and even construct a *continuous* semiflow on a *compact* space with an invariant probability measure such that the original semigroup is (isomorphic to) the Koopman semigroup induced by this continuous semiflow, compare [1]. We call this a *topological model* of the measurable semiflow. For this terminology (in the time discrete case) see [5], Chapter 12. The proof is based on techniques used there.

**Theorem 3.1** (Topological Model). *Let  $(X, \Sigma, \mu)$  be a probability space and  $(T(t))_{t \geq 0}$  a strongly continuous Markov lattice semigroup on  $L^1(X)$  with generator  $A$ . Then there exists a compact space  $K$ , a continuous semiflow  $(\psi_t)_{t \geq 0}$  on  $K$  and a  $(\psi_t)_{t \geq 0}$ -invariant Borel probability measure  $\nu$  such that the Koopman semigroup  $(T(t))_{t \geq 0}$  on  $L^1(X)$  is isomorphic to the Koopman semigroup induced by  $(\psi_t)_{t \geq 0}$  on  $L^1(K, \nu)$ .*

*Proof.* By 2.1 we know that the semigroup preserves  $L^\infty(X)$  and its generator is a derivation on the subalgebra  $D(A) \cap L^\infty(X)$ . We define a suitable  $C^*$ -subalgebra in  $L^\infty(X)$  to translate the problem into a  $C(K)$  situation,  $K$  compact.

Consider  $\mathcal{A} := \{f \in L^\infty(X) : s \mapsto T(s)f \text{ is } \|\cdot\|_\infty\text{-continuous}\}$ . This is an algebra with  $\mathbf{1} \in \mathcal{A}$  since for  $f, g \in \mathcal{A}$  we have  $f \cdot g \in \mathcal{A}$ , because of the multiplicativity of  $(T(t))_{t \geq 0}$ . Since  $\mathcal{A}$  is closed with respect to  $\|\cdot\|_\infty$  and closed under conjugation, it is a commutative  $C^*$ -algebra. Clearly,  $(T(t))_{t \geq 0}$  leaves  $\mathcal{A}$  invariant. We show

that  $\mathcal{A}$  is dense in  $L^1(X)$  with respect to  $\|\cdot\|_1$ . The strong continuity of  $(T(t))_{t \geq 0}$  on  $L^1(X)$  implies that  $\|\cdot\|_1\text{-}\lim_{t \searrow 0} \frac{1}{t} \int_0^t T(r)f \, dr = f$  for each  $f \in L^\infty(X)$ . So it remains to show that  $\int_0^t T(r)f \, dr \in \mathcal{A}$  for  $f \in L^\infty(X)$ . Indeed, for  $0 \leq s \leq t$ , using that Markov lattice operators are  $\|\cdot\|_\infty$ -isometries, we obtain

$$\begin{aligned} \left| T(s) \int_0^t T(r)f \, dr - \int_0^t T(r)f \, dr \right| &= \left| \int_0^t T(s+r)f \, dr - \int_0^t T(r)f \, dr \right| \\ &= \left| \int_s^{t+s} T(r)f \, dr - \int_0^t T(r)f \, dr \right| \\ &\leq \left| \int_t^{t+s} T(r)f \, dr \right| + \left| \int_0^s T(r)f \, dr \right| \\ &\leq 2s\|f\|_\infty \mathbf{1}. \end{aligned}$$

This shows that  $s \mapsto T(s) \int_0^t T(r)f \, dr$  is  $\|\cdot\|_\infty$ -continuous at zero, hence on  $[0, \infty)$ . So  $\frac{1}{t} \int_0^t T(r)f \, dr \in \mathcal{A}$  for all  $t \geq 0$  and  $f \in \mathcal{A}$ . Therefore,  $\overline{\mathcal{A}}^{\|\cdot\|_1} = L^1(X)$ . By the Theorem of Gelfand-Naimark (cf. [5], Chapter 4, Section 4) there is a  $C^*$ -isomorphism  $\Phi: \mathcal{A} \rightarrow C(K)$  for a suitable compact space  $K$ . Since  $(T(t))_{t \geq 0}$  leaves  $\mathcal{A}$  invariant, [2], Chapter BII.3, Theorem 3.4 yields a  $C_0$ -semigroup on  $C(K)$  of the form  $S(t)g = g \circ \psi_t$  for  $g \in C(K)$ , where  $(\psi_t)_{t \geq 0}$  is a continuous semiflow on  $K$ . By the Representation Theorem of Riesz we conclude that  $\int_X \Phi^{-1}g \, d\mu = \int_K g \, d\nu$  for all  $g \in C(K)$  and for a unique probability measure  $\nu$  on  $K$ . This implies that  $\Phi$  can be extended to an isometry from  $L^1(X, \mu)$  onto  $L^1(K, \nu)$  and the assertion follows.  $\square$

**Corollary 3.2.** *If, in the situation of 3.1, the measure space is separable, then the compact space  $K$  can be chosen to be metrizable.*

*Proof.* Since the measure space is separable there is a countable number of measurable sets  $\{M_1, M_2, \dots\}$  generating the  $\sigma$ -algebra  $\Sigma$ . Then every integral of the form  $\int_0^t T(s)\mathbf{1}_{M_k} \, ds$  belongs to the above algebra  $\mathcal{A}$ . We consider

$$\mathcal{M} := \left\{ \int_0^{t_n} T(s)\mathbf{1}_{M_k} \, ds : k, n \in \mathbb{N} \right\},$$

with  $(t_n)_{n \in \mathbb{N}}$  dense in  $[0, \infty)$ . Then every  $\mathbf{1}_{M_k}$  lies in  $\overline{\mathcal{M}}^{\|\cdot\|_1}$ . Therefore, the closure of the linear hull  $\overline{\text{lin}}(\mathcal{M})$  is  $\|\cdot\|_1$ -dense in  $L^1(X)$ . Now take  $\mathcal{A}(\mathcal{M})$  as the closed algebra generated by  $\mathcal{M}$ . This is a separable  $(T(t))_{t \geq 0}$ -invariant  $C^*$ -algebra with respect to  $\|\cdot\|_\infty$  which is  $\|\cdot\|_1$ -dense in  $L^1(X)$ . Hence, it is isomorphic to a separable  $C(K)$  space by the Gelfand Naimark Theorem, where  $K$  is metrizable (see Theorem 4.7 in [5]).  $\square$

## 4. ERGODIC FLOWS

We now start from a measurable semiflow  $(\varphi_t)_{t \geq 0}$  on a measure space  $(X, \Sigma, \mu)$  and its corresponding Koopman semigroup  $(T(t))_{t \geq 0}$  on  $L^1(X)$ . We recall the following properties.

**Definition 4.1.** *Let  $(X, \Sigma, \mu)$  be a probability space.*

1. *A measurable measure-preserving semiflow  $(\varphi_t)_{t \geq 0}$  on  $X$  is called ergodic if the only measurable  $(\varphi_t)_{t \geq 0}$ -invariant sets are trivial.*
2. *A Koopman semigroup  $(T(t))_{t \geq 0}$  on  $L^p(X)$ ,  $1 \leq p < \infty$ , is called irreducible if there are no nontrivial closed  $(T(t))_{t \geq 0}$ -invariant ideals in  $L^p(X)$ .*

For a semiflow  $(\varphi_t)_{t \geq 0}$  and its induced Koopman semigroup  $(T(t))_{t \geq 0}$  as in 2.2 the two properties are equivalent. (cf. [5], Chapter 4, Theorem 4.8 and Chapter 7, Exercise 8). Obviously, if some  $\varphi_t$  is ergodic, so is the semiflow  $(\varphi_t)_{t \geq 0}$ . The opposite is not true in general. This situation is described in [9], Theorem 1 for which we now give a short proof using results from the Perron Frobenius spectral theory.

**Theorem 4.2.** *Let  $(\varphi(t))_{t \geq 0}$  be an ergodic measurable measure-preserving semiflow on a standard probability space  $(X, \Sigma, \mu)$  that induces a Koopman semigroup  $(T(t))_{t \geq 0}$  on  $L^2(X)$  with generator  $(A, D(A))$ . Then at most a countable number of mappings  $\varphi_t$ ,  $t > 0$ , are not ergodic.*

*Proof.* Let  $(\varphi_t)_{t \geq 0}$  be ergodic, hence  $(T(t))_{t \geq 0}$  be irreducible. By [2], Section C-III, Theorem 3.8 we know that the boundary point spectrum  $G := P\sigma(A) \cap i\mathbb{R}$  is a group. Since  $L^2(X)$  is separable,  $G$  must be countable since eigenvectors corresponding to different eigenvalues are orthogonal.

We know that for  $t > 0$  fixed,  $\varphi_t$  is not ergodic if and only if  $\dim \text{fix}(T(t)) \geq 2$ . By the spectral mapping theorem for the point spectrum (cf. [7], Section 3, Theorem 3.7) we have

$$\text{fix}(T(t)) = \overline{\text{lin}} \left\{ f \in L^2(X) \left| \begin{array}{l} Af = i\lambda f, \lambda \in \mathbb{R} \text{ such that} \\ i\lambda t = 2\pi i k \text{ for } k \in \mathbb{Z} \end{array} \right. \right\}.$$

Since  $i\lambda$  belongs to the countable group  $G$ , there is only a countable number of  $t_k$  such that  $\dim \text{fix}(T_{t_k}) \geq 2$ .  $\square$

The assertion in 4.2 does not hold for non separable spaces. Take for example the infinite torus  $\mathbb{T}^{(0, \infty)} := \{(x_r)_{r > 0} : x_r \in \mathbb{T}\}$  and the rotation

$$\begin{aligned} \varphi_t : \mathbb{T}^{(0, \infty)} &\rightarrow \mathbb{T}^{(0, \infty)} \\ (x_r)_{r > 0} &\mapsto (e^{2\pi i t r} \cdot x_r)_{r > 0}. \end{aligned}$$

Then the fixed space of the induced Koopman semigroup is one dimensional, while  $\dim \text{fix}(T(t)) \geq 2$  for all  $t > 0$ .

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